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A FORMULA FOR THE REACTIONS OF IDEAL CONSTRAINTS†

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A formula for determining the reactions of Lagrange-ideal constraints when using arbitrary linear quasi-velocities is obtained. An example of rolling of a sphere along a pair of skew lines is considered with different assumptions regarding whether stick or slip occurs at the points of contact. © 2003 Elsevier Science Ltd. All rights reserved.

As a rule, in the mechanics of systems with ideal constraints, the forms of the equations of motion are used in which the reactions of the constraints do not occur explicitly. This is a considerable advantage of the Lagrange equations of the second kind, Chaplygin's equations, the Boltzmann–Hamel equations etc. However, if the system is subjected to unilateral constraints, some of which may disappear (decrease) or, conversely, appear (be imposed) during the motion of the system, then in practice the only method of establishing whether a constraint is imposed on the system is to constantly check the condition for the reaction of this constraint to be equal to zero. In other words, the values of the reactions of all unilateral constraints in systems of variable structure must be calculated during the motion.

It is easy to obtain a formula for the reactions of ideal constraints when the equations of the reactions are independent (see, for example, [1]). This formula was also obtained in [2, 3] for the case when there is an excessive number of constraint equations, when written in the form of second-order equations, they form a set that is linearly dependent on the generalized accelerations. The use of an excessive number of constraint equations will often be justified in order not to disturb the symmetry of these equations and the system of variables occurring in them [4–6].

Below, the formula derived by Kalaba *et al.* in [2, 3] is extended to the case when arbitrary linear quasi-velocities are used in systems with ideal constraints, and is obtained by a somewhat different method having a clearer mechanical meaning.

1. A LINEAR SYSTEM

Suppose A is an arbitrary real $m \times n$ matrix of rank $r > 0$. The following assertion is further required: the general solution of a consistent system of linear equations $Ax = b(x \in R^n)$ can be represented in the form

$$x = A^+b + Gy, \quad G = E - A^+A \tag{1.1}$$

Here A^+ is the pseudo-inverse $n \times m$ matrix [7], E is the identity matrix and $y \in R^n$ is an arbitrary vector.

There are several proofs of this formula [2, 8, 9].

We will present one more proof. The set of solutions of the inhomogeneous system is closed, and when $b \neq 0$ the norm of these solutions is strictly positive. Consequently, a solution exists, that is minimum in norm, which, as is well known [7], has the form A^+b. It remains to prove that rank $G = n - r$, i.e. the second term in (1.1) is the most general solution of the homogeneous system $Ax = 0$. By the property of the pseudo-inverse matrix

$$AG = A - AA^+A = A - A = 0$$

Hence, rank $G \leqslant n - r$. On the other hand, the matrix A^+A is always symmetrical, and its rank is not greater than r, since the rank of the pseudo-inverse matrix is equal to the rank of the initial matrix. A non-singular matrix W

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exists such that $W^{-1}GW$ is a diagonal matrix. But then there are not less than $n - r$ diagonal elements (ones) in this matrix. Consequently, rank $G = n - r$.

Effective numerical methods exists for calculating the pseudo-inverse matrix. It can also be written in analytical form. Suppose $A = BC$ is a skeleton representation of the initial matrix [7]. Then

$$A^+ = C'(CC')^{-1}(B'B)^{-1}B'$$

(the prime denotes transposition).

One more formula is required for pseudo-inverse matrices

$$D^+ = D'(DD')^+ \tag{1.2}$$

To prove this we will consider the consistent system of linear equations $Dx = b$, when D is an arbitrary non-zero $m \times n$ matrix, and $x \in R^n$. The condition for this system to be non-contradictory has the form (see (1.1))

$$DD^+b = b \tag{1.3}$$

A particular solution of the system that is minimum in norm is $x_p = D^+b$.

Consider another system $DD'z = b$, where $D'z = x$. Its solution, that is minimum in norm, is $z_p = (DD')^+b$, whence, by virtue of the uniqueness of the particular solution of the initial linear system that is minimum in norm, its follows that

$$(D^+ - D'(DD')^+)b = 0 \tag{1.4}$$

But condition (1.3) is the only limitation on the vector b. and hence expression (1.2) follows from (1.4).

2. A FORMULA FOR THE REACTIONS

Suppose a mechanical system with a finite number of degrees of freedom is described by the vector of local generalized coordinates $q \in R^n$. Lagrange-ideal kinematic constraints

$$a_i(t, q, \dot{\pi}) = 0, \quad i = 1, \ldots, m$$

are imposed on the system, where t is the time and $\dot{\pi}$ is the n-vector of linear quasi-velocities, introduced using the non-singular $n \times n$ matrix P and the free vector $h \in R^n$

$$\dot{\pi} = P(t, q)\dot{q} + h(t, q), \quad \det P \neq 0$$

Taking the total derivative with respect to time of the constraint equations, we obtain

$$A(t, q, \dot{\pi})\ddot{\pi} = b(t, q, \dot{\pi}) \tag{2.1}$$

We will write the equations of motion of the mechanical system in Euler–Lagrange form [10] with multipliers $\lambda \in R^m$ of the constraints (2.1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\pi}_j} - \frac{\partial T}{\partial \pi_j} + \sum_{k=1}^{n}\sum_{r=1}^{n} \gamma_{rj}^k \frac{\partial T}{\partial \dot{\pi}_k}\dot{\pi}_r + \sum_{k=1}^{n} \alpha_j^k \frac{\partial T}{\partial \dot{\pi}_k} = P_j + \sum_{i=1}^{m} \lambda_i a_{ij}$$

$$j = 1, \ldots, n$$

Here T is the kinetic energy of the system, P_j are the generalized active forces applied to the system $A = \|a_{ij}\|$, and formulae for the multi-index symbols γ_{rj}^k and α_j^k are derived in [10].

We can write these equations in matrix form as follows:

$$M\ddot{\pi} = F(t, q, \dot{\pi}) + A'\lambda \tag{2.2}$$

where M is the positive definite symmetrical $n \times n$ matrix of the form T_2 of the kinetic energy of the system, and all terms of the Euler–Lagrange equations which do not contain quasi-accelerations (apart from the covector $A'\lambda$ of the reaction of constraints (2.1)) are collected in the expression F.

Note that $a = M^{-1}F$ is the vector of the quasi-accelerations of the system, freed from constraints (2.1).

In order to determine the reactions of the constraints, we will eliminate the vector of the quasi-accelerations $\ddot{\pi}$ from formulae (2.1) and (2.2) and, introducing the notation,

$$B = AM^{-1/2}, \quad B'\lambda = x, \quad b - Aa = c$$

we obtain the matrix equation

$$Bx = c \ (\mathrm{rank} B = \mathrm{rank} A = r)$$

In the general case the matrix B is rectangular.

According to expression (1.1), the general solution of this equation has the form

$$x = B^+c + (E - B^+B)y, \quad y \in R^n$$

Consequently

$$A'\lambda = F_1 + F_2, \quad F_1 = M^{1/2}B^+c, \quad F_2 = M^{1/2}(E - B^+B)y \tag{2.3}$$

The covectors F_1 and F_2 on the right-hand side of the last formula are mutually orthogonal

$$M^{-1}F_2' \cdot F_1 = y'(E - B^+B)M^{-1/2} \cdot F_1 = y'(B^+ - B^+BB^+)c = y'(B^+ - B^+)c = 0$$

On the other hand, the vector $M^{-1}F_2$ belongs to the set of virtual displacements of the system

$$A\delta q = 0$$

In fact

$$A \cdot M^{-1}F_2 = B(E - B^+B)y = 0$$

But, by convention, the constraints (2.1) imposed on the system are ideal; consequently, $y = 0$ and we obtain from formula (2.3) the expression

$$A'\lambda = M^{1/2}(AM^{-1/2})^+(b - Aa) \tag{2.4}$$

for the reaction of these constraints.

Formula (2.4) was derived somewhat differently in [2, 3] for mechanical systems specified in generalized coordinates. A general analogue of this formula for systems with Lagrange non-ideal constraints has also been published in [11].

Note that we can avoid having to calculate the matrix $M^{1/2}$ in expression (2.4). That is, using property (1.2) of the pseudo-inverse matrix for $D = AM^{-1/2}$ we obtain

$$(AM^{-1/2})^+ = M^{-1/2}A'(AM^{-1}A')^+$$

and formula (2.4) takes the form

$$A'\lambda = A'(AM^{-1}A')^+(b - Aa) \tag{2.5}$$

which is more convenient for practical applications.

When the matrix A has maximum rank, equal to $\min(m, n)$, expression (2.5) simplifies to

$$A'\lambda = A'(AM^{-1}A')^{-1}(b - Aa) \tag{2.6}$$

3. EXAMPLE

As an example which illustrates the explicit expressions obtained for the reaction of constraints, we will consider the problem of the motion of a heavy uniform sphere of radius R and mass m, supported by a pair of skew lines.

We will choose a fixed Cartesian system of coordinates $Oxyz$. Suppose $OO' = a$ is the distance between the lines, and the point O lies on the first line of support. We will direct the Oy axis along the section OO', while the Oz axis is perpendicular to the plane containing the section OO' and the second line of support. Hence this axis makes an acute angle α with the ascending vertical. We will denote by β the smallest of the two angles between the skew lines, and by (X, Y, Z) the coordinates of the centre C of the sphere.

The conditions for the sphere to be in contact with both lines of support have the form

$$\begin{aligned} &[(Y-a)^2+Z^2]^{1/2}-R=0, \\ &[Y^2+(X\sin\beta-Z\cos\beta)^2]^{1/2}-R=0 \end{aligned} \tag{3.1}$$

We will consider three cases of the interaction between the sphere and the lines of support.

In system S_1 the sphere rolls without sliding. The conditions for the instantaneous velocities of the sphere at points of contact C_1 and C_2 to be zero are expressed by the equations

$$v_C+\omega\times\rho_i=0, \quad \rho_i=CC_i, \quad i=1,2 \tag{3.2}$$

where v_C is the velocity of the centre of the sphere and $\omega(p, q, r)$ is the angular velocity.

In system S_2 the sphere rolls without sliding along the first line of support and slides without friction along the second line. The constraint equations are (3.1) and Eq. (3.2) with $i = 1$.

Finally, in system S_4 the sphere slides without friction along both lines of support.

We will further derive some results of numerical calculations in graphical form, obtained for the following values of the parameters

$$m=1, R=2, a=3, \beta=\pi/6, \alpha=\pi/12$$

The initial conditions were assumed to be the same in all versions of the calculations: $X_0 = 2.5$ and $\omega_0 = -1.55$. The sphere begins to move without sliding, resting on both lines and having an initial velocity of rotation ω_0 around the instantaneous axis C_1C_2 in the direction OO'.

We will put

$$\dot{\pi}_1=\dot{X}, \quad \dot{\pi}_2=\dot{Y}, \quad \dot{\pi}_3=\dot{Z}, \quad \dot{\pi}_4=p, \quad \dot{\pi}_5=q, \quad \dot{\pi}_6=r$$

System S_1 (the holonomic system integrated in quadratures with one degree of freedom). There are eight constraint equations (3.1) and (3.2). Converting them to the form (2.1), we obtain, using formula (2.5), an explicit expression for the generalized reaction force – the right-hand side of the formula

$$\left\| \begin{matrix} R_1+R_2 \\ \rho_1\times R_1+\rho_2\times R_2 \end{matrix} \right\| = A'\lambda \tag{3.3}$$

where R_1 and R_2 are the forces acting on the sphere from the first and second lines of support respectively.

The sphere, acted upon by the force of gravity and the reactions, will rotate as long as the following two inequalities are satisfied

$$\rho_1\cdot R_1\le 0, \quad \rho_2\cdot R_2\le 0 \tag{3.4}$$

However, it is impossible to verify these conditions in system S_1 since, the unknowns R_1 and R_2 can only be found from Eqs (3.3) to within the vector kC_1C_2, where k is an arbitrary scalar. Consequently, the settings of the kinematic constraints (3.2) are insufficient for the dynamic determinability of the system, and additional conditions are necessary.

In Fig. 1 we show a curve of the values of the projection of the instantaneous angular velocity of the sphere on the direction C_1C_2 as a function of the coordinate X of the centre of the sphere. The curve was obtained by numerical integration of equations of motion (2.2) of system S_1.

The point A on the curve corresponds to the initial position of the sphere. The sphere begins to move to the origin of coordinates $X_0 = 0$ and, being barely able to reach it, stops, and begins to roll backwards with an ever increasing angular velocity. In Fig. 1 the direction around the curve is denoted by the arrow. At the extreme right-hand point of this curve the distance $C_1C_2 = 2R$, and the sphere continues to roll along the skew lines of support but from below. After reaching the extreme left-hand point of the curve

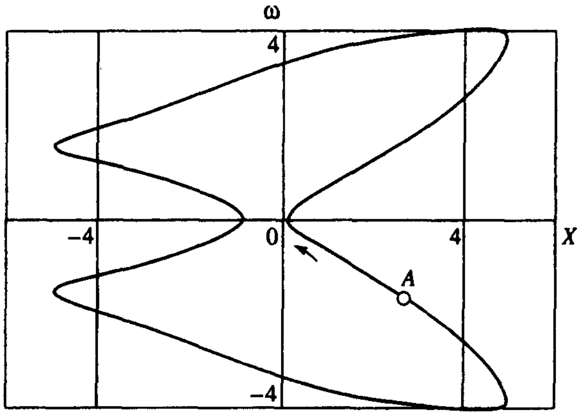

Fig. 1

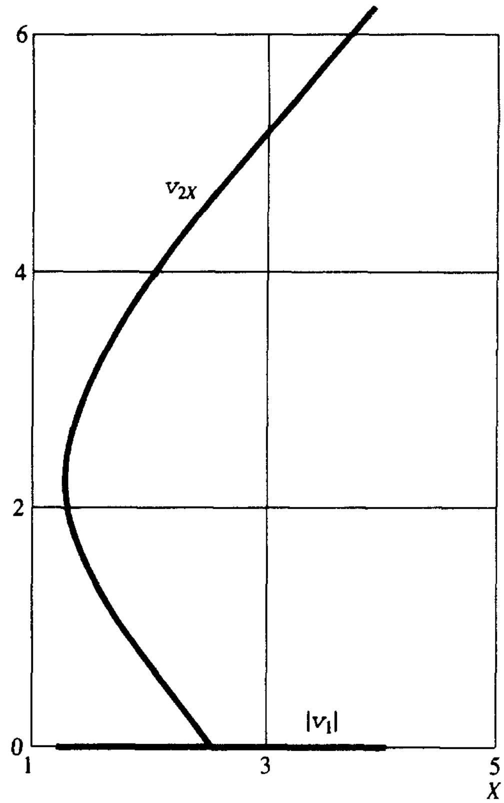

Fig. 2

in Fig. 1, at which the distance between the support points is equal to the diameter, the sphere "emerges" on top and, slowing down, begins to approach the origin of coordinates. At $X = -0.89$ the sphere stops, and then begins to roll in the opposite direction, until it reaches the initial position A. The motion of the sphere, in accordance with Eqs (2.2) and the formula obtained for the generalized reaction, is periodic. However, it is physically unreal, since constraints (3.1) and (3.2) are not bilateral.

System S_2 (the non-holonomic system with two degrees of freedom). There are five constraint equations (3.1) and (3.2) for $i = 1$. Formula (2.5) with the corresponding matrices A and b for system S_2 gives an explicit expression for the generalized reaction (3.3). On the left-hand side of this formula $R_2 = \mu_2\rho_2$. The sphere does not leave both support lines, so long as the following inequalities are satisfied

$$\rho_1 \cdot R_1 \le 0, \quad \mu_2 \le 0 \qquad (3.5)$$

The unknowns R_1 and μ_2 are found from Eqs (3.3).

In Fig. 2 we show graphs of the values of the velocities of the points C_1 and C_2 of the sphere. The modulus of the instantaneous velocity of the point C_1 is equal to zero during the whole time of motion ($|v_1| = 0$, the thickened section of the X axis), while the projection of the instantaneous velocity of the point C_2 of the sphere onto the OX axis (the curve v_{2x}) increases rapidly, when the sphere moves away from the extreme position $X = 1.25$.

Figure 3 gives a representation of the reaction forces acting on the sphere. In the motion considered, the contact with the first support line disappears first (the graph reaches the axis $\rho_1 \cdot R_1 = 0$).

System S_4 (the holonomic system integrable in quadratures with four degrees of freedom). The equations of its constraints (3.1) are independent, and the generalized reaction is calculated from formula (2.6). In this case both forces $R_1 = \mu_1\rho_1$ and $R_2 = \mu_2\rho_2$ are found from Eqs (3.3), while the conditions for the sphere to remain in contact with both supports take the form $\mu_1 \leqslant 0$, $\mu_2 \leqslant 0$.

Figure 4 shows graphs of the functions $\mu_1(X)$ and $\mu_2(X)$. The sphere also ceases to move from the first line of support, but for a value of X somewhat greater than in the similar situation in system S_2.

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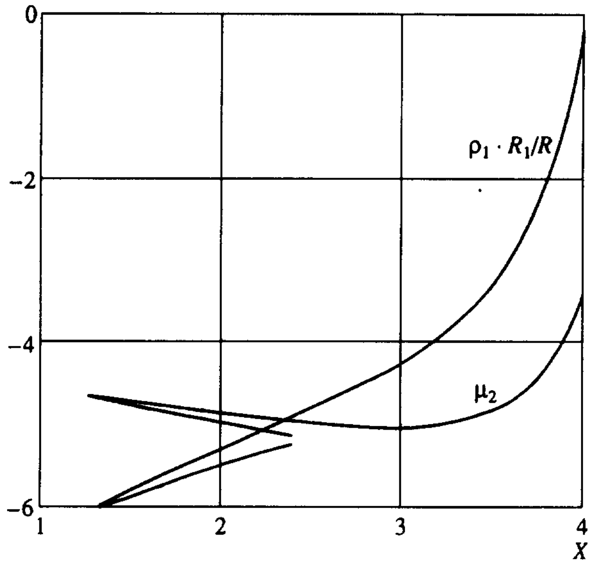

Fig. 3

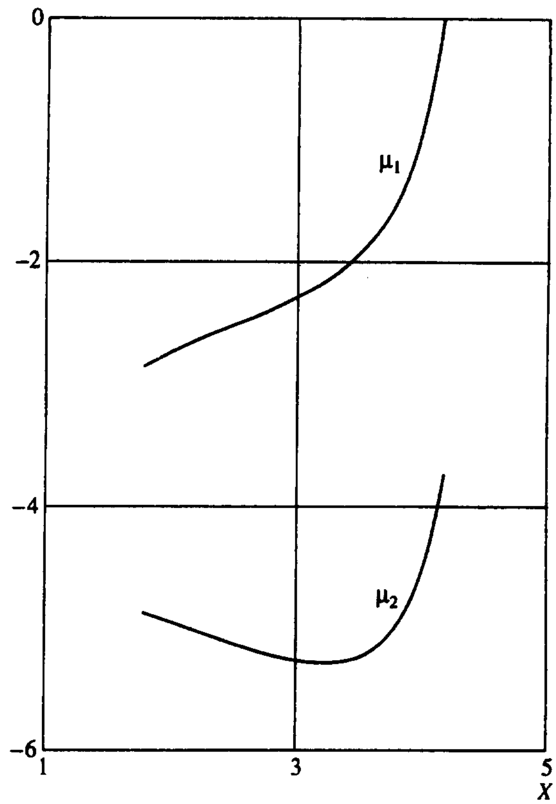

Fig. 4

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